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1993 J. Phys. A: Math. Gen. 26 6101

(http://iopscience.iop.org/0305-4470/26/22/013)

J. Phys. A: Math. Gen. 26 (1993) 6101–6113. Printed in the UK

Dynamical systems for quasiperiodic chains and new self-similar polynomials

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Received 14 June 1993

Abstract. Dynamical systems in $SL(2, \mathbb{R})$ or $SL(2, \mathbb{C})$ naturally appear in the transfer matrix method for quasiperiodic chains characterized by arbitrary irrational numbers. We show new sub-dynamical systems and invariants that are related to full diagonal and off-diagonal components of the transfer matrices; they are analogous to formulae of Chebyshev polynomials of the first and second kinds. Applying them to an electronic problem on the Fibonacci chain, we obtain sets of self-similar polynomials, *quasiperiodic extension of the Chebyshev polynomials* of the first and second kinds with self-similar properties. Two scaling factors of the self-similarities coincide with ones obtained by the perturbative decimation renormalization group method.

1. Introduction

A dynamical system in $SL(2, \mathbb{R})$ or $SL(2, \mathbb{C})$ defined by

$$M_{k+1} = M_{k-1} M_k \tag{1.1}$$

called 'the Fibonacci sequence in a Lie group' appears in the transfer matrix study of the quasiperiodic Fibonacci chain model [1, 2]. A key feature of (1.1) is the existence of a sub-dynamical system and an invariant for $F_k = \operatorname{tr} M_k$,

$$F_{k+2} = F_{k+1} F_k - F_{k-1} \tag{1.2}$$

and

$$I = F_{k+1}^2 + F_k^2 + F_{k-1}^2 - F_{k+1} F_k F_{k-1} - 2\,. \tag{1.3}$$

This sub-dynamical system is called the 'trace map', which is enough for spectrum study of physical problems (electronic, magnetic, phonon) and makes clear a singular, continuous-spectrum Cantor set with Lebesgue measure zero [1, 3–5]. It is known that eigenfunctions are critical according to the singular continuous spectrum [6].

In this paper, we first clarify why the above invariants and others of the same type can exist. Second, we present new complementary sub-dynamical systems and new invariants of (1.1) associated not only with traces, but also with full diagonal and off-diagonal components of the matrices [7], which directly relate to how the components of the matrices grow, namely the

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0305-4470/93/226101+13$07.50

behaviour of critical wavefunctions on electronic problems [8]. Moreover, third, we construct new sets of 'self-similar polynomials' [9] using the new sub-dynamical systems.

To obtain a deeper understanding of the trace map, Dotera has developed the theory of 'self-similar polynomials' through the observation that a transfer-type tight-binding model on the Fibonacci chain plays a key role for the dynamics. As a result, an important self-similar behaviour of the polynomials has been found. It has been argued that the polynomials are powerful tools to elucidate the structure of the energy spectrum of any types of models on the chain [10].

In the theory, F_k's are polynomials of $x = \operatorname{tr} M_1$ and are considered as 'quasiperiodic extensions' of the Chebyshev polynomials of the *first* kind possessing self-similar behaviour with two scaling factors. In this paper we present new sets of self-similar polynomials G_k's and H_k's for our new sub-dynamical systems as quasiperiodic extensions of the Chebyshev polynomials of the *second* kind.

These self-similar polynomials have one parameter r representing the strength of quasiperiodicity. Properties of these self-similar polynomials are the following. (i) The order of these polynomials are Fibonacci numbers. (ii) When $r = 1$ (the periodic case), they coincide with the Chebyshev polynomials of the first and second kinds. (iii) When $|r| \ll 1$ (a strong quasiperiodic region), they exhibit self-similarities with two scaling factors; 'seed polynomials' ($k = 1, 2, 3$) appear self-similarly. (iv) They have invariants which are independent of x. (v) The seed polynomials can be generated from their approximate forms.

In terms of these self-similar polynomials, we uncover a remarkable hidden self-similar characteristic of the dynamical system (1.1), since the two scaling factors in the self-similarities are nothing but those obtained by a decimation RG method [11], which has been considered as a different approach to the trace-map method.

The paper is organized as follows. In section 2, we begin to study generic quasiperiodic chains constructed by the projection method [12] associated with arbitrary irrational numbers. (The Fibonacci chain corresponds to a typical quadratic irrational called the golden mean.) It is easy to see the relation between self-similarity of the structure of chains and quadratic irrationality.

In section 3, we introduce the transfer matrix method for the chains which provides generic dynamical systems [13], i.e. generalization of (1.1). We investigate full components of the matrices and obtain new sub-dynamical systems governed by the orbit of the trace map and new invariants. The new maps and invariants are described in terms of analogous formulae of the Chebyshev polynomials. Using the knowledge of section 2, we demonstrate that the invariants of maps exist for quasiperiodic chains with any arbitrary irrationals because of a theorem of the 'free group of rank 2'.

In section 4, we restrict our attention to the dynamical system for the Fibonacci chain and construct sets of self-similar polynomials. Maps, invariants and relations are explicitly computed in detail. The conclusion is given in section 5. In appendix A, an outline of derivation of the new maps and invariants is given, and in appendix B, the self-similar polynomials up to the sixth Fibonacci generation are shown.

2. Quasiperiodic chains associated with arbitrary irrationals

We can obtain a quasiperiodic lattice by projecting lattice points contained in a subset of a higher-dimensional space into a lower-dimensional subspace. This method is called the projection method and the subset is called a 'window' [12]. Using a projection method with the 'window' $\{(x_1, x_2) \in \mathbb{R}^2 | \quad \alpha x_1 < x_2 \leqslant \alpha x_1 + \alpha + 1\}$, where $\alpha > 0$ is an irrational number, and regarding the long (short) tile as A and the short (long) one as B when $\alpha < (>)1$, we define a quasiperiodic chain (sequence) $R(\alpha)$ consisting of two elements A and B. Let q_k/p_k be a rational approximant of α given by

$$\frac{q_k}{p_k} = n_0 + \frac{1|}{|n_1} + \frac{1|}{|n_2} + \cdots + \frac{1|}{|n_{k-1}} \tag{2.1}$$

where

$$q_{-1} = 0 \qquad q_0 = 1 \qquad q_{k+1} = n_k q_k + q_{k-1} \quad k \geqslant 0 \tag{2.2}$$

$$p_{-1} = 1 \qquad p_0 = 0 \qquad p_{k+1} = n_k p_k + p_{k-1} \quad k \geqslant 0\,. \tag{2.3}$$

We introduce a Nielsen transformation of $\mathcal{F}_2$ (the free group of rank 2 generated by A and B) [14], $\chi_n : \mathcal{F}_2 \times \mathcal{F}_2 \to \mathcal{F}_2 \times \mathcal{F}_2$:

$$\chi_n(X, Y) = (X^n Y, X) \tag{2.4}$$

and define a finite sequence R_k by

$$(R_0, R_{-1}) = (B, A) \qquad (R_{k+1}, R_k) = \chi_{n_k}(R_k, R_{k-1}) \qquad \text{i.e.} \quad R_{k+1} = R_k^{n_k} R_{k-1} \quad k \geqslant 0\,. \tag{2.5}$$

It holds that, for $k \geqslant 2$

$$R_k = R(\alpha)|_{s_k} = \begin{cases} B^{\alpha_0}AB^{\alpha_1}A\ldots B^{\alpha_{p_k-1}}A = A^{\tilde\alpha_0}BA^{\tilde\alpha_1}B\ldots A^{\tilde\alpha_{q_k-1}}BA & \text{if } k \text{ is odd} \\ B^{\alpha_0}AB^{\alpha_1}A\ldots B^{\alpha_{p_k-1}}AB = A^{\tilde\alpha_0}BA^{\tilde\alpha_1}B\ldots A^{\tilde\alpha_{q_k-1}}B & \text{if } k \text{ is even} \end{cases} \tag{2.6}$$

and

$$R_k \to R(\alpha) \qquad \text{as} \qquad k \to \infty \tag{2.7}$$

where

$$\alpha_l = [(l+1)\alpha] - [l\alpha] \qquad \text{and} \qquad \tilde\alpha_l = [(l+1)/\alpha] - [l/\alpha] \tag{2.8}$$

and $R(\alpha)|_{s_k}$ denotes the first s_k elements of $R(\alpha)$ with $s_k = p_k + q_k$ [15, 16].

It also holds that generators of the group of automorphisms of $\mathcal{F}_2$

$$\sigma_1 : A \mapsto B \qquad \text{and} \qquad B \mapsto A \tag{2.9}$$

$$\sigma_2 : A \mapsto BA \qquad \text{and} \qquad B \mapsto B \tag{2.10}$$

$$\sigma_3 : A \mapsto A \qquad \text{and} \qquad B \mapsto B^{-1} \tag{2.11}$$

induce actions by generators of $GL(2, \mathbb{Z})$

$$R(\alpha) \mapsto R\left(\frac{1}{\alpha}\right) = R\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \alpha\right) = R(\rho_1 \cdot \alpha) \tag{2.12}$$

$$R(\alpha) \mapsto R(\alpha + 1) = R\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \alpha\right) = R(\rho_2 \cdot \alpha) \tag{2.13}$$

$$R(\alpha) \mapsto R(-\alpha) = R\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \alpha\right) = R(\rho_3 \cdot \alpha) \tag{2.14}$$

respectively, with some natural extension of the definition of $R(\alpha)$. These generators generate generic 'hyperinflation transformations, generalizations of those obtained in [17].

Let us mention the relation between quadratic irrationality of α and self-similarity of $R(\alpha)$. Hereafter we assume $0 < \alpha < 1$ $(n_0 = 0)$ by virtue of the symmetry (2.12). Since a quadratic irrational is a fixed point of a certain $GL(2, \mathbb{Z})$ transformation, the corresponding chain is invariant under a certain hyperinflation transformation, meaning that the chain is self-similar. If we define a free substitution (an automorphism) [14]

$$\phi_n = \sigma_1 {\sigma_2}^n \qquad \phi_n : A \mapsto A^n B \qquad \text{and} \qquad B \mapsto A \tag{2.15}$$

we get

$$\begin{aligned} R_k &= \phi_{n_1} \circ \phi_{n_2} \circ \cdots \circ \phi_{n_{k-1}}(A) \\ &= \phi_{n_1} \circ \phi_{n_2} \circ \cdots \circ \phi_{n_{k-1}} \circ \phi_{n_k}(B) \\ R_{k+1} &= \phi_{n_1} \circ \phi_{n_2} \circ \cdots \circ \phi_{n_{k-1}} \circ \phi_{n_k}(A) \end{aligned} \tag{2.16}$$

which means that R_k and R_{k+1} are obtained by transforming B and A by the same automorphism (notice the order of operating). Thus it is clear that, if α has a periodic continued fraction expansion, $R(\alpha)$ can be obtained from A (or B) by an infinite succession of a certain free substitution (substitution rule) and have self-similarity. (The continued fraction expansion of an irrational number is periodic if and only if it is quadratic irrational.) In particular, if $n_k = n$, i.e. independent of k, the chains are the Fibonacci (golden) and the precious (silver, etc) mean chains [18]. They are obtained by an infinite succession of ϕ_n and are the 'fixed points' of the substitution ϕ_n, i.e. have self-similar properties.

3. The dynamical systems and their sub-dynamical systems

Many physical problems on the chain $R(\alpha)$ can be reduced to the transfer matrix method. From (2.5), the matrix M_k satisfies

$$M_{k+1} = M_{k-1} M_k^{n_k} \tag{3.1}$$

which defines a dynamical system in $SL(2, \mathbb{R})$ or $SL(2, \mathbb{C})$. Traces of the matrices were well studied by several authors [1–5, 13, 18, 19]. However, our purpose here is to investigate other components of the matrices, which provide new sub-dynamical systems and invariants of (3.1).

Let us define the following quantities using an arbitrary $P \in M(2, \mathbb{C})$ with $\operatorname{tr} P = 0$:

$$F_k = \operatorname{tr} M_k \qquad G_k = \operatorname{tr}(P M_k) = \operatorname{tr}(P M_{k-2} M_{k-1}^{n_{k-1}}) \qquad H_k = \operatorname{tr}(M_{k-2} P M_{k-1}^{n_{k-1}}) . \tag{3.2}$$

In order to see the relations of F_k, G_k and H_k, we also write down the equations of F_k throughout the paper. Then, we obtain formulae

$$\begin{aligned} F_{k+2} S_{n_k}(F_k) + F_{k-1} S_{n_{k+1}}(F_{k+1}) &= F_{k+1} F_k S_{n_{k+1}}(F_{k+1}) S_{n_k}(F_k) \\ &\quad - F_{k+1} S_{n_{k+1}}(F_{k+1}) S_{n_k-1}(F_k) - F_k S_{n_k}(F_k) S_{n_{k+1}-1}(F_{k+1}) \end{aligned} \tag{3.3}$$

$$\begin{aligned} G_{k+2} S_{n_k}(F_k) - G_{k-1} S_{n_{k+1}}(F_{k+1}) &= F_{k+1} G_k S_{n_{k+1}}(F_{k+1}) S_{n_k}(F_k) \\ &\quad + G_{k+1} S_{n_{k+1}}(F_{k+1}) S_{n_k-1}(F_k) - G_k S_{n_k}(F_k) S_{n_{k+1}-1}(F_{k+1}) \end{aligned} \tag{3.4}$$

$$\begin{aligned} H_{k+2} S_{n_k}(F_k) + G_{k-1} S_{n_{k+1}}(F_{k+1}) &= G_{k+1} F_k S_{n_{k+1}}(F_{k+1}) S_{n_k}(F_k) \\ &\quad - G_{k+1} S_{n_{k+1}}(F_{k+1}) S_{n_k-1}(F_k) - G_k S_{n_k}(F_k) S_{n_{k+1}-1}(F_{k+1}) \end{aligned} \tag{3.5}$$

where $S_n(x)$ is the monic Chebyshev polynomial of the second kind defined by

$$S_0(x) = 0 \qquad S_1(x) = 1 \qquad S_{n+1}(x) = xS_n(x) - S_{n-1}(x)$$
$$\text{or} \qquad S_n(2\cos\theta) = \frac{\sin n\theta}{\sin\theta} \tag{3.6}$$

and $C_n(x)$ is the monic Chebyshev polynomial of the first kind defined by

$$C_0(x) = 2 \qquad C_1(x) = x \qquad C_{n+1}(x) = xC_n(x) - C_{n-1}(x)$$
$$\text{or} \qquad C_n(2\cos\theta) = 2\cos n\theta\,. \tag{3.7}$$

It is immediately shown that

$$M_kM_{k+1}M_k^{-1}M_{k+1}^{-1} = (M_{k-1}M_kM_{k-1}^{-1}M_k^{-1})^{-1} = (M_0M_1M_0^{-1}M_1^{-1})^{(-1)^k} \tag{3.8}$$

and therefore there exist invariants (i.e. quantities independent of k)

$$I = \mathrm{tr}\,(M_{k-1}M_kM_{k-1}^{-1}M_k^{-1})$$
$$= 2 + \tfrac{1}{2}\mathrm{tr}\,([M_{k-1}, M_k]_-{}^2) = 2 - \det[M_{k-1}, M_k]_- = \det[M_{k-1}, M_k]_+ - 2 \tag{3.9}$$
$$J = (-1)^k\mathrm{tr}\,(PM_{k-1}M_kM_{k-1}^{-1}M_k^{-1}) \tag{3.10}$$

where $[M, N]_\pm = MN \pm NM$, satisfying

$$S_{n_k}(F_k)^2(I+2) = F_{k+1}^2 + F_k^2S_{n_k}(F_k)^2 + F_{k-1}^2 - F_{k+1}F_kF_{k-1}S_{n_k}(F_k) + 2F_{k+1}F_{k-1}S_{n_k-1}(F_k)$$
$$= F_{k+1}^2 + F_k^2S_{n_k}(F_k)^2 + F_{k-1}^2 - F_{k+1}F_{k-1}C_{n_k}(F_k) \tag{3.11}$$
$$S_{n_k}(F_k)^2(-1)^kJ = F_{k+1}G_{k+1} - F_kG_kS_{n_k}(F_k)^2 + F_{k-1}G_{k-1}$$
$$- F_{k+1}F_kG_{k-1}S_{n_k}(F_k) + (F_{k+1}G_{k-1} + G_{k+1}F_{k-1})S_{n_k-1}(F_k)$$
$$= -F_{k+1}H_{k+1} - F_kG_kS_{n_k}(F_k)^2 + F_{k-1}G_{k-1}$$
$$+ F_{k+1}G_kF_{k-1}S_{n_k}(F_k) - (F_{k+1}G_{k-1} - G_{k+1}F_{k-1})S_{n_k-1}(F_k)\,. \tag{3.12}$$

(An outline of derivation of these formulae is given in appendix A.) These formulae provide sub-dynamical systems and invariants of (3.1).

To see this explicitly, we set

$$P^{(a)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad P^{(b)} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad P^{(c)} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{3.13}$$

Then we obtain that $G_k^{(a)}$, $G_k^{(b)}$, and $G_k^{(c)}$ satisfy these formulae (with obvious notations) and M_k can be described in the form

$$M_k = \begin{pmatrix} (F_k + G_k^{(a)})/2 & G_k^{(b)} \\ G_k^{(c)} & (F_k - G_k^{(a)})/2 \end{pmatrix} \tag{3.14}$$

which implies that our sub-dynamical systems completely determine the original system. Moreover, we know how the components of the matrix grow from (3.4) [8]. Since the sub-dynamical system (3.4) is linear about G_k, the orbits of matrices are governed by the orbit of their traces. It is clear that we can obtain four independent invariants of (3.1) and that in generic cases P can be expanded by $P^{(a)}$, $P^{(b)}$ and $P^{(c)}$.

The very existence of the invariants is due to the relations (2.16) and (3.8). The dynamical system (3.1) defines a representation of $\mathcal{F}_2$ in $SL(2,\mathbb{R})$ or $SL(2,\mathbb{C})$, corresponding A to M_1 and B to M_0, in general R_k to M_k. As mentioned in section 2, R_k and R_{k+1} are obtained from A and B by the same automorphism, and there exists a theorem of group theory that the commutator of generators $\mathcal{F}_2$ is transformed as

$$ABA^{-1}B^{-1} \mapsto X(ABA^{-1}B^{-1})^{\pm 1}X^{-1} \tag{3.15}$$

(X is an element of $\mathcal{F}_2$) by an automorphism [14]. Therefore the commutator of M_k and M_{k+1} must satisfy the same relation as (3.15), which is (3.8). It is plausible that we cannot obtain such invariants as I or J for quasiperiodic chain obtained by non-free substitution rules (not automorphisms) [20, 21]. No analogue of this theorem exists for any $\mathcal{F}_m$ ($m \geqslant 3$), hence it is also plausible that we cannot obtain such invariants for m tiling chains ($m \geqslant 3$).

Formulae (3.3)–(3.12) can be regarded as quasiperiodic analogues of those of the Chebyshev polynomials of the first and the second kinds. In fact, if $M_1 = M_0$, i.e. $M_k = M_1^{s_k}$ (the chain is periodic), then we find F_k's an G_k's are the Chebyshev polynomials of $x = \operatorname{tr} M_1$, and $I = 2$, $J = 0$. Since the maps defined by the Chebyshev polynomials have ergodic properties (bounded in a region), we can understand that a periodic system has an absolutely continuous spectrum in the view of the dynamical system [22]. In contrast, quasiperiodic systems have singular continuous spectra in general, a scenario which has been recognized in terms of escaping in the trace map (3.3) and the invariant (3.11) [3, 13, 19].

Before closing this section, we return to the Fibonacci chain, $n_k = 1$ (α becomes the inverse of the golden mean (0.618...), a typical quadratic irrational number). In this case, the chain is self-similar and obtained by a substitution rule. The dynamical system (3.1) becomes (1.1), which is thought to be a renormalization group equation by virtue of the self-similarity of the chain [1, 2, 23]. The sub-dynamical systems and the invariants in this case become

$$F_{k+2} = F_{k+1}F_k - F_{k-1} \tag{3.16}$$

$$G_{k+2} = F_{k+1}G_k + G_{k-1} \tag{3.17}$$

$$H_{k+2} = G_{k+1}F_k - G_{k-1} \tag{3.18}$$

and

$$I = F_{k+1}^2 + F_k^2 + F_{k-1}^2 - F_{k+1}F_kF_{k-1} - 2 \tag{3.19}$$

$$\begin{aligned} J &= (-1)^k(F_{k+1}G_{k+1} - F_kG_k + F_{k-1}G_{k-1} - F_{k+1}F_kG_{k-1}) \\ &= (-1)^k(-F_{k+1}H_{k+1} - F_kG_k + F_{k-1}G_{k-1} + F_{k+1}G_kF_{k-1}). \end{aligned} \tag{3.20}$$

4. Sets of self-similar polynomials

In the rest of the paper, we concentrate on the dynamical system (1.1) for the Fibonacci chain and present new sets of self-similar polynomials $G_k(x)$'s and $H_k(x)$'s corresponding to the Chebyshev polynomials of the *second* kind concerning (3.17) and (3.18).

In the transfer-type tight-binding model on the Fibonacci chain [9], we have

$$M_1 = \begin{pmatrix} x & -1 \\ 1 & 0 \end{pmatrix} \qquad M_2 = \begin{pmatrix} x & -1/r \\ 1 & 0 \end{pmatrix}\begin{pmatrix} rx & -r \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} rx^2 - 1/r & -rx \\ rx & -r \end{pmatrix} \tag{4.1}$$

where $r = t_A/t_B$ is the ratio of the transfer energies and we set $x = E/t_A$ (E is the energy value).

In order to construct self-similar polynomials $G_k(x)$'s and $H_k(x)$'s, we must make a suitable choice of P. If we make a correct choice of

$$P = M_1 \begin{pmatrix} -1 & 2r^2x \\ 0 & 1 \end{pmatrix} M_1^{-1} \tag{4.2}$$

then we have seed polynomials ($k = 1, 2, 3$)

$$F_1(x) = x \qquad F_2(x) = rx^2 - r - 1/r \qquad F_3(x) = rx^3 - (2r + 1/r)x \tag{4.3}$$

$$G_1(x) = (2r^2 - 1)x \qquad G_2(x) = rx^2 + r - 1/r \qquad G_3(x) = (2r^3 - r)x^3 + (-2r^3 + 1/r)x \tag{4.4}$$

$$H_1(x) = x \qquad H_2(x) = (2r^3 - r)x^2 + 1/r - r \qquad H_3(x) = rx^3 - x/r\,. \tag{4.5}$$

The following properties of the polynomials $G_k(x)$'s and $H_k(x)$'s are obtained.

(a) When $r = 1$ (the periodic case), $G_k(x)$'s and $H_k(x)$'s coincide with the Chebyshev polynomials of the second kind, i.e.

$$F_k(x) = C_{f_k}(x) \qquad G_k(x) = H_k(x) = xS_{f_k}(x) \tag{4.6}$$

where f_k is the Fibonacci number ($f_0 = f_1 = 1$, $f_{k+1} = f_k + f_{k-1}$).

(b) When $|r| \ll 1$ (a strong quasiperiodic region), $G_k(x)$'s and $H_k(x)$'s exhibit self-similar behaviour, as the $F_k(x)$'s.

(c) J is independent of x, i.e. independent of the energy value; we obtain

$$I = r^2 + 1/r^2 \qquad J = r^2 - 1/r^2\,. \tag{4.7}$$

(d) Seed polynomials can be generated from their approximate forms.

Many choices of P lead to (a)–(c) but not (d), as is plausible from our studies.

Now we will compute (b) and (d) (see appendix B). We find self-similar properties in two ways: 3-cycles and 2-cycles.

(A) *3-cycles*, for $|x| < |r| \ll 1$; self-similarities with a scaling factor $1/r^2$ exist:

$$\begin{array}{lll} F_4(x) \simeq F_1(x/r^2) & F_5(x) \simeq -F_2(x/r^2) & F_6(x) \simeq -F_3(x/r^2) \\ G_4(x) \simeq -G_1(x/r^2) & G_5(x) \simeq G_2(x/r^2) & G_6(x) \simeq G_3(x/r^2) \\ H_4(x) \simeq -H_1(x/r^2) & H_5(x) \simeq H_2(x/r^2) & H_6(x) \simeq H_3(x/r^2)\,. \end{array} \tag{4.8}$$

Moreover, if we employ the approximate seed functions

$$\begin{array}{lll} \tilde{F}_1(x) = x & \tilde{F}_2(x) = -r - 1/r & \tilde{F}_3(x) = -x/r \\ \tilde{G}_1(x) = -x & \tilde{G}_2(x) = r - 1/r & \tilde{G}_3(x) = x/r \\ \tilde{H}_1(x) = x & \tilde{H}_2(x) = 1/r - r & \tilde{H}_3(x) = -x/r \end{array} \tag{4.9}$$

which satisfy $\tilde{I} = r^2 + 1/r^2$ and $\tilde{J} = r^2 - 1/r^2$, then from (3.16)–(3.18) we obtain the exact relations

$$\begin{aligned}
&\tilde{F}_4(x) = F_1(x/r^2) && \tilde{F}_5(x) = -F_2(x/r^2) && \tilde{F}_6(x) = -F_3(x/r^2) \\
&\tilde{G}_4(x) = -G_1(x/r^2) && \tilde{G}_5(x) = G_2(x/r^2) && \tilde{G}_6(x) = G_3(x/r^2) \\
&\tilde{H}_4(x) = -H_1(x/r^2) && \tilde{H}_5(x) = H_2(x/r^2) && \tilde{H}_6(x) = H_3(x/r^2)\,.
\end{aligned} \tag{4.10}$$

(B) *2-cycles*, for $|x| < |r| \ll 1$; other self-similarities with a scaling factor $2/r$ exist:

$$\begin{aligned}
&F_3(x \pm R) \simeq F_1(2x/r) && F_4(x \pm R) \simeq \pm F_2(2x/r) && F_5(x \pm R) \simeq \pm F_3(2x/r) \\
&G_3(x \pm R) \simeq G_1(2x/r) && G_4(x \pm R) \simeq \pm G_2(2x/r) && G_5(x \pm R) \simeq \pm G_3(2x/r) \\
&H_3(x \pm R) \simeq H_1(2x/r) && H_4(x \pm R) \simeq \pm H_2(2x/r) && H_5(x \pm R) \simeq \pm H_3(2x/r)
\end{aligned} \tag{4.11}$$

where $R = r + 1/r$. Furthermore, if we use the approximate seed functions in the vicinity of $\pm R$

$$\begin{aligned}
&\bar{F}_1(x) = \pm(r + 1/r) && \bar{F}_2(x) = \pm 2x && \bar{F}_3(x) = 2x/r \\
&\bar{G}_1(x) = \pm(r - 1/r) && \bar{G}_2(x) = \pm 2x && \bar{G}_3(x) = (4r - 2/r)x \\
&\bar{H}_1(x) = \pm(r + 1/r) && \bar{H}_2(x) = \mp 2x && \bar{H}_3(x) = 2x/r
\end{aligned} \tag{4.12}$$

which satisfy $\bar{I} = r^2 + 1/r^2$ and $\bar{J} = r^2 - 1/r^2$, then we obtain the exact relations

$$\begin{aligned}
&\bar{F}_3(x) = F_1(2x/r) && \bar{F}_4(x) = \pm F_2(2x/r) && \bar{F}_5(x) = \pm F_3(2x/r) \\
&\bar{G}_3(x) = G_1(2x/r) && \bar{G}_4(x) = \pm G_2(2x/r) && \bar{G}_5(x) = \pm G_3(2x/r) \\
&\bar{H}_3(x) = H_1(2x/r) && \bar{H}_4(x) = \pm H_2(2x/r) && \bar{H}_5(x) = \pm H_3(2x/r)\,.
\end{aligned} \tag{4.13}$$

These mean that not only (b) but also (d) is fulfilled.

These self-similarities for $F_k(x)$'s and $G_k(x)$'s are illustrated in figure 1. Seed functions are shown in (a) and (d). Global structures appear again for three cycles in (b) and (e), and for 2-cycles in (c) and (e). The energy value x is scaled by two scale factors, which coincide with the ones obtained by the perturbative decimation renormalization group method [11]. Although new $G_k(x)$'s and $H_k(x)$'s are not directly related to the decimation transformation, it is important that the scaling factors are recovered due to the specific choice of P (the meaning of the choice of P is still to be determined, though).

The two-way self-similarity is well described by the schema of the 'Fibonacci tree' [9, 10]. The Fibonacci chain is obviously self-similar in the sense of the inflation rule; however, the the property is more subtle, a fusion of two-way self-similarity. According to (b) or (d), self-similar polynomials are akin to fixed points of the maps (3.16)–(3.18), in other words, 'fixed-point functions' under scale transformations in the case of the strong quasiperiodicity. Weak quasiperiodic case where $r \simeq 1$ has been investigated in [10].

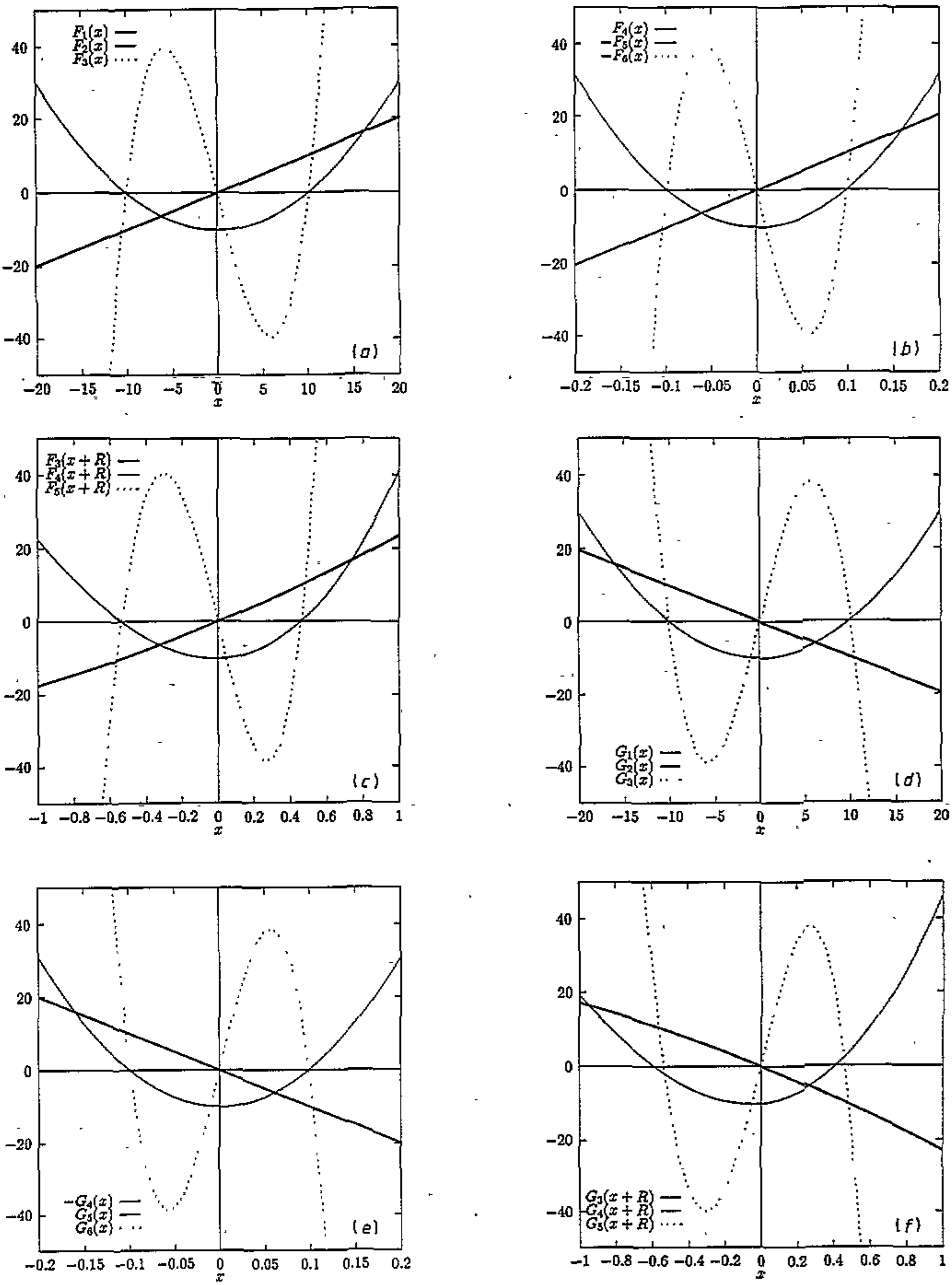

Figure 1. Self-similar polynomials with $r = \frac{1}{10}$. Global structures appear again in self-similar manners. (*a*) Seed functions, $F_1(x)$, $F_2(x)$, $F_3(x)$. (*b*) Magnification 100× in the x direction of $F_4(x)$, $-F_5(x)$, $-F_6(x)$. (*c*) Magnification 20× in the x direction of $F_3(x+R)$, $F_4(x+R)$, $F_5(x+R)$. (*d*) Seed functions, $G_1(x)$, $G_2(x)$, $G_3(x)$. (*e*) Magnification 100× in the x direction of $-G_4(x)$, $G_5(x)$, $G_6(x)$. (*f*) Magnification 20× in the x direction of $G_3(x+R)$, $G_4(x+R)$, $G_5(x+R)$.

5. Conclusion

In this paper, we investigated dynamical systems associated with quasiperiodic chains and found new sub-dynamical systems and invariants. A key feature we pointed out was that they were analogous to formulae of the Chebyshev polynomials of the first and second kinds. Our new dynamical systems directly represent how the components of the matrices grow. Moreover, we presented sets of self-similar polynomials associated with the Fibonacci chain and illustrated two kinds of self-similarity for the chain. By means of a parameter r representing the strength of quasiperiodicity, these polynomials were quite a natural quasiperiodic extension of the Chebyshev polynomials of the first and the second kinds.

There exists a difference between the sets of the polynomials regarding inner product properties. $F_k(x)$'s are not mutually orthogonal with respect to their zeros (the Chebyshev polynomials of the first kind are orthogonal); however, their inner products can be exactly represented by r and the Fibonacci and modified Fibonacci numbers as explored by Dotera [9]. $G_k(x)$'s and $H_k(x)$'s were not studied for this property, since the Chebyshev polynomials of the second kind do not have the property.

Quasiperiodic chains are models of quasicrystals [24] and have many attractive aspects from not only physical but also mathematical points of view. The issue of the dynamical system through the transfer matrix method for the quasiperiodic chain is one of them. The transfer matrix method gives a simple mathematical form, whereas the decimation renormalization group method [11, 25] gives a simple physical picture; both theories give consistent results without a clear relation between them. Apparently the self-similar polynomials provide a substantial understanding of the relation and are an analytic realization of the self-similarity of the Fibonacci chain.

Acknowledgments

We would like to thank R Abe, A Hatano, J Matuzawa, Y Takahashi, N Tokushige, and P J Steinhardt for encouragement and useful conversations. TD is supported in part by the National Science Foundation, Material Research Laboratories Program, under Grant No DMR-8519059, during stay at University of Pennsylvania (1990–92).

Appendix A

An outline of derivation of formulae (3.3)–(3.5) and (3.11), (3.12) is given.

Let $M, N \in SL(2,\mathbb{R})$ or $SL(2,\mathbb{C})$, $x = \operatorname{tr} M$, $\bar{x} = \operatorname{tr}(PM)$, $y = \operatorname{tr} N$ with $P \in M(2,\mathbb{C})$ satisfying $\operatorname{tr} P = 0$. Then, we have the following:

$$M^n = S_n(x)M - S_{n-1}(x)E \qquad \text{and} \qquad M^{-n} = -S_n(x)M + S_{n+1}(x)E \tag{A.1}$$

$$M + M^{-1} = xE \qquad \text{and} \qquad N + N^{-1} = yE \tag{A.2}$$

where E is the unit matrix. Using (A.2), we have

$$\operatorname{tr} M = \operatorname{tr}(M^{-1}) \qquad \text{and} \qquad \operatorname{tr}(PM) = -\operatorname{tr}(PM^{-1}) \tag{A.3}$$

$$\operatorname{tr}(MN) = xy - \operatorname{tr}(MN^{-1}) = xy - \operatorname{tr}(NM^{-1}) \tag{A.4}$$

$$\operatorname{tr}(PMN) = \bar{x}y - \operatorname{tr}(PMN^{-1}) = \bar{x}y + \operatorname{tr}(PNM^{-1}) . \tag{A.5}$$

From (3.1), we have $M_{k+2} = M_k M_{k+1}^{n_{k+1}}$, $M_{k-1} = M_{k+1} M_k^{-n_k}$. Thus, using (A.1) we obtain

$$M_{k+2} = S_{n_{k+1}}(F_{k+1}) M_k M_{k+1} - S_{n_{k+1}-1}(F_{k+1}) M_k \tag{A.6}$$

$$M_{k-1} = -S_{n_k}(F_k) M_{k+1} M_k + S_{n_k+1}(F_k) M_{k+1} \tag{A.7}$$

$$P M_{k+2} = S_{n_{k+1}}(F_{k+1}) P M_k M_{k+1} - S_{n_{k+1}-1}(F_{k+1}) P M_k \tag{A.8}$$

$$P M_{k-1} = -S_{n_k}(F_k) P M_{k+1} M_k + S_{n_k+1}(F_k) P M_{k+1} \tag{A.9}$$

$$P M_{k-1}^{-1} = S_{n_k}(F_k) P M_k M_{k+1}^{-1} - S_{n_k-1}(F_k) P M_{k+1}^{-1} \tag{A.10}$$

$$M_k P M_{k+1}^{n_{k+1}} = S_{n_{k+1}}(F_{k+1}) M_k P M_{k+1} - S_{n_{k+1}-1}(F_{k+1}) P M_k . \tag{A.11}$$

To get the trace formula concerning F_k, we take traces of (A.6) and (A.7) and eliminate $\mathrm{tr}\,(M_k M_{k+1}) = \mathrm{tr}\,(M_{k+1} M_k)$. Then, we get (3.3). In order to obtain the trace formula concerning G_k, we take traces of (A.8) and (A.10). Noticing (A.3) we have

$$G_{k+2} = S_{n_{k+1}}(F_{k+1}) \mathrm{tr}\,(P M_k M_{k+1}) - S_{n_{k+1}-1}(F_{k+1}) G_k \tag{A.12}$$

$$-G_{k-1} = S_{n_k}(F_k) \mathrm{tr}\,(P M_k M_{k+1}{}^{-1}) + S_{n_k-1}(F_k) G_{k+1} . \tag{A.13}$$

Using (A.5), we find

$$\mathrm{tr}\,(P M_k M_{k+1}) = F_{k+1} G_k - \mathrm{tr}\,(P M_k M_{k+1}^{-1}) . \tag{A.14}$$

Then, (A.12)–(A.14) follow (3.4). To show the trace formula concerning H_k, we take traces of (A.9) and (A.11) and eliminate $\mathrm{tr}\,(P M_k M_{k+1}) = \mathrm{tr}\,(M_{k+1} P M_k)$. Then, (3.5) is shown.

Moreover, using (A.3)–(A.5), we have

$$\begin{aligned} I &= \mathrm{tr}\,(M_{k-1} M_k M_{k-1}^{-1} M_k^{-1}) = \{\mathrm{tr}\,(M_{k-1} M_k)\}^2 - \mathrm{tr}\,(M_{k-1} M_k^2 M_{k-1}) \\ &= \{\mathrm{tr}\,(M_{k-1} M_k)\}^2 - \mathrm{tr}\, M_{k-1} \cdot \mathrm{tr}\,(M_k^2 M_{k-1}) + \mathrm{tr}\,(M_k^2) \\ &= \{\mathrm{tr}\,(M_{k-1} M_k)\}^2 - \mathrm{tr}\, M_{k-1} \cdot \{\mathrm{tr}\, M_k \cdot \mathrm{tr}\,(M_k M_{k-1}) - \mathrm{tr}\, M_{k-1}\} + \{\mathrm{tr}\, M_k\}^2 - 2 \\ &= \{\mathrm{tr}\,(M_{k-1} M_k)\}^2 + \{\mathrm{tr}\, M_k\}^2 + \{\mathrm{tr}\, M_{k-1}\}^2 - \mathrm{tr}\,(M_{k-1} M_k) \cdot \mathrm{tr}\, M_k \cdot \mathrm{tr}\, M_{k-1} - 2 \end{aligned} \tag{A.15}$$

and

$$\begin{aligned} (-1)^k J &= \mathrm{tr}\,(P M_{k-1} M_k M_{k-1}^{-1} M_k^{-1}) \\ &= \mathrm{tr}\,(P M_{k-1} M_k) \cdot \mathrm{tr}\,(M_{k-1} M_k) - \mathrm{tr}\,(P M_{k-1} M_k^2 M_{k-1}) \\ &= \mathrm{tr}\,(P M_{k-1} M_k) \cdot \mathrm{tr}\,(M_{k-1} M_k) - \mathrm{tr}\,(P M_{k-1}) \cdot \mathrm{tr}\,(M_k^2 M_{k-1}) - \mathrm{tr}\,(P M_k^2) \\ &= \mathrm{tr}\,(P M_{k-1} M_k) \cdot \mathrm{tr}\,(M_{k-1} M_k) \\ &\quad - \mathrm{tr}\,(P M_{k-1}) \cdot \{\mathrm{tr}\, M_k \cdot \mathrm{tr}\,(M_k M_{k-1}) - \mathrm{tr}\, M_{k-1}\} - \mathrm{tr}\,(P M_k) \cdot \mathrm{tr}\, M_k \\ &= \mathrm{tr}\,(P M_{k-1} M_k) \cdot \mathrm{tr}\,(M_{k-1} M_k) - \mathrm{tr}\,(P M_k) \cdot \mathrm{tr}\, M_k + \mathrm{tr}\,(P M_{k-1}) \cdot \mathrm{tr}\, M_{k-1} \\ &\quad - \mathrm{tr}\,(M_{k-1} M_k) \cdot \mathrm{tr}\, M_k \cdot \mathrm{tr}\,(P M_{k-1}) . \end{aligned} \tag{A.16}$$

In addition, from (A.6) and (A.8),

$$F_{k+1} = S_{n_k}(F_k) \mathrm{tr}\,(M_{k-1} M_k) - S_{n_k-1}(F_k) F_{k-1} \tag{A.17}$$

and

$$G_{k+1} = S_{n_k}(F_k) \mathrm{tr}\,(P M_{k-1} M_k) - S_{n_k-1}(F_k) G_{k-1} \tag{A.18}$$

are obtained. Eliminating $\mathrm{tr}\,(M_{k-1} M_k)$ and $\mathrm{tr}\,(P M_{k-1} M_k)$ from (A.15)–(A.18), we obtain (3.11) and (3.12).

Appendix B

Self-similar polynomials up to the sixth Fibonacci order are shown:

$$F_1(x) = x$$

$$F_2(x) = rx^2 - (r + 1/r)$$

$$F_3(x) = rx^3 - (2r + 1/r)x$$

$$F_4(x) = r^2x^5 - (3r^2 + 2)x^3 + (2r^2 + 2 + 1/r^2)x$$

$$\begin{aligned} F_5(x) = {} & r^3x^8 - (5r^3 + 3r)x^6 + (8r^3 + 9r + 3/r)x^4 \\ & - (4r^3 + 7r + 4/r + 1/r^3)x^2 + (r + 1/r) \end{aligned}$$

$$\begin{aligned} F_6(x) = {} & r^5x^{13} - (8r^5 + 5r^3)x^{11} + (25r^5 + 30r^3 + 10r)x^9 \\ & - (38r^5 + 66r^3 + 42r + 10/r)x^7 + (28r^5 + 64r^3 + 59r + 26/r + 5/r^3)x^5 \\ & - (8r^5 + 25r^3 + 32r + 19/r + 6/r^3 + 1/r^5)x^3 + (2r^3 + 6r + 4/r + 1/r^3)x \end{aligned}$$

$$G_1(x) = (2r^2 - 1)x$$

$$G_2(x) = rx^2 + r - 1/r$$

$$G_3(x) = (2r^3 - r)x^3 - (2r^3 - 1/r)x$$

$$G_4(x) = r^2x^5 - (r^2 + 2)x^3 + x/r^2$$

$$\begin{aligned} G_5(x) = {} & (2r^5 - r^3)x^8 - (8r^5 + r^3 + 3r)x^6 + (10r^5 + 6r^3 - 3r - 3r^2)x^4 \\ & - (4r^5 + 4r^3 - r - 2/r - 1/r^3)x^2 + (r - 1/r) \end{aligned}$$

$$\begin{aligned} G_6(x) = {} & r^5x^{13} - (6r^5 + 5r^3)x^{11} + (13r^5 + 22r^3 + 10r)x^9 \\ & - (12r^5 + 32r^3 + 30r + 10/r)x^7 + (4r^5 + 16r^3 + 27r + 18/r + 5/r^3)x^5 \\ & + (r^3 - 8r - 9/r - 4/r^3 - 1/r^5)x^3 - (2r^3 - 2/r - 1/r^3)x \end{aligned}$$

$$H_1(x) = x$$

$$H_2(x) = (2r^3 - r)x^2 - r + 1/r$$

$$H_3(x) = rx^3 - x/r$$

$$H_4(x) = (2r^4 - r^2)x^5 - (4r^4 + r^2 - 2)x^3 + (2r^4 - 1/r^2)x$$

$$\begin{aligned} H_5(x) = {} & r^3x^8 - (3r^3 + 3r)x^6 + (2r^3 + 5r + 3/r)x^4 \\ & - (r + 2/r + 1/r^3)x^2 - r + 1/r \end{aligned}$$

$$\begin{aligned} H_6(x) = {} & (2r^7 - r^5)x^{13} - (14r^7 + 2r^5 - 5r^3)x^{11} + (38r^7 + 27r^5 - 10r^3 - 10r)x^9 \\ & - (50r^7 + 60r^5 + 6r^3 - 22r - 10/r)x^7 \\ & + (32r^7 + 52r^5 + 22r^3 - 15r - 16/r - 5/r^3 + 1/r^5)x^5 \\ & - (8r^7 + 16r^5 + 15r^3 - 4r - 9/r - 4/r^3)x^3 + (4r^3 - 2/r - 1/r^3)x\,. \end{aligned}$$

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